

Solid Continuum with Thermofluctuation Kinetics of Microcracks. Phase Transition.

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Abstract

Thermodynamic equations for a solid and a solid continuum under stress are derived on the basis of a multicomponent mean field Markov process for thermofluctuation kinetics of microcracks. The resulting continuum is viscous elastoplastic continuum with damage. It can radiate elastic waves. The existence of phase transitions with microcrack density as an order parameter is proved for a stationary state of a special model of solid. For a finite large system the distribution of the logarithmic power of acoustic emission at a critical point is similar to the distribution of the logarithmic energy of earthquakes.

INTRODUCTION

This paper presents a system of equations for the non-equilibrium dynamics of an elastic continuum involving the appearance and healing of microcracks. The basis for deriving the equations is a multicomponent continuous-time mean field Markov process with intensities of the activation type. Since the contribution of a crack to the deformation of the body is memorized after the healing, the resulting behavior is nonlinear viscous elastoplasticity. This type of behavior demonstrates qualitative correspondence with the real properties of a solid. For instance, a solid shows brittleness and elasticity at low temperatures and/or large rates of deformation and fluidity and plasticity at high temperatures and/or small rates of deformation.

For simplest model the existence of phase transition between phases of a low and high density of microcracks is strictly proved. An area of the high density phase is a soft inclusion in the continuum as its pliability is increased. For certain types of stress field this area can propagate in the continuum like a crack and radiate elastic waves. Therefore this continuum can serve as model for earthquake generation.

This system of equations expands the range of known equations for continuous continuum such as the gas dynamics equations, Navier–Stokes equations etc.

The idea of using the activation principle in crack kinetics dates back to the kinetic concept of strength suggested by S.N.Zhurkov (**author?**) [Zhurkov 1965] The empirical Zhurkov formula

$$\tau = \tau_0 \exp \left\{ \frac{U - \nu \sigma}{k_B T} \right\} \quad (0.1)$$

describes lifetime τ of a specimen under tensile load σ at temperature T , where k_B is Boltzman's constant, τ_0 , U and ν are constants. In spite of the fact that Zhurkov's formula clearly indicates the thermofluctuation mechanism of fracture, numerous attempts to create on this basis a mathematical apparatus of the theory have not been successful. The reason seems to lie in the fact that the quantity $1/\tau$ cannot be directly used as intensity of microfracture generation, Firstly, τ is a macroscopic quantity and cannot serve as "a first principle" but must be calculated from equations known beforehand. Secondly, the energy in the numerator of the expression in the braces must be a quadratic, but not a linear, function of σ , because the linearity contradicts the definition of elastic energy.

However, if we use an expression of the type $1/\tau$ with quadratic dependence on the stress as the intensity of microcrack appearance, physically correct equations can then be derived. Moreover, as will be demonstrated below, experimental relations of specimen lifetime τ versus σ and T presented by Zhurkov as (0.1) are reproduced in our model..

1. MULTICOMPONENT MEAN FIELD PROCESS

Definition 1. Let $\bar{\xi}_N(t) \equiv \{\xi_N(x, t), x \in \Omega_N\}$, $\xi_N(x, t) = 0, 1$, $|\Omega_N| = N$, $t \geq 0$, ($|A|$ denotes the number of elements in A) be a N -component continuous-time Markov process with state-space $\{0, \dots, K\}^{\Omega_N}$. Denote $\mathbf{y}(t) = \left\{ \frac{n_1(t)}{N}, \dots, \frac{n_K(t)}{N} \right\}$, $n_k(t) \equiv n_k(\bar{\xi}_N(t)) = |\{x \in \Omega_N : \xi_N(x, t) = k\}|$, $k = 1, \dots, K$. and let $\Lambda_k(\mathbf{z})$, $M_k(\mathbf{z})$, $\mathbf{z} = (z_1, \dots, z_K)$, be positive continuous functions on $[0, 1]^K$. We assume that only point translations $\{0 \rightarrow k, k \rightarrow 0\}$, $k = 1, \dots, K$, are possible with the rates given by conditional probabilities

$$\begin{cases} \Pr\{\xi_N(x, t+h) = k, k > 0 | \xi_N(x, t) = 0, \mathbf{y}(t)\} = \Lambda_k(\mathbf{y}(t))h + o(h), \\ \Pr\{\xi_N(x, t+h) = 0 | \xi_N(x, t) = k, k > 0, \mathbf{y}(t)\} = M_k(\mathbf{y}(t))h + o(h) \end{cases} \quad (1.1)$$

(As usual $\Pr\{A\}$ denotes the probability of A and $\Pr\{A|B\}$ denotes the conditional probability of A given B .) This process will be called a "mean field" Markov process.

Below we are going to prove that components $\xi_N(x, t)$ of stationary Markov process $\bar{\xi}_N$ with $K = 1$ are asymptotically independent at $N \rightarrow \infty$ (see Appendix)..

For stationary processes with $K \geq 2$ and non-stationary processes $\bar{\xi}(t) \equiv \bar{\xi}_\infty(t)$ the proof of the independence of components is unavailable. Therefore we need the additional definition in these cases..

Definition 2. A mean field Markov process $\bar{\xi} \equiv \{\xi(x, t), x \in \Omega, t \geq 0\}$ on a countable set of points Ω is a set of independent Markov processes $\xi(x, t), x \in \Omega$, with state-space $\{0, \dots, K\}$, with conditional probabilities

$$\begin{cases} \Pr\{\xi_N(x, t+h) = k, k > 0 | \xi_N(x, t) = 0, \mathbf{y}(t)\} = \Lambda_k(\mathbf{p}(t))h + o(h), \\ \Pr\{\xi_N(x, t+h) = 0 | \xi_N(x, t) = k, k > 0, \mathbf{y}(t)\} = M_k(\mathbf{p}(t))h + o(h) \end{cases}$$

where $\mathbf{p}(t) = \{p_k(t), k = 1, \dots, K\}$, $p_k(t) \equiv \Pr\{\xi(x, t) = k\}$.

Lemma. The probabilities $p_k(t)$ are described by equations

$$\frac{dp_k(t)}{dt} = \left[1 - \sum_{i=1}^K p_i(t) \right] \Lambda_k(\mathbf{p}(t)) - p_k(t) M_k(\mathbf{p}(t)), \quad k = 1, \dots, K.$$

Proof. According to the law of large numbers $\lim_{N \rightarrow \infty} \frac{n_k(t)}{N} = p_k(t)$, $k = 1, \dots, K$ and the equations are easily deduced from 1.1 by the following manipulation:

$$\begin{aligned} p_k(t+h) &= \sum_{k'=0, \dots, K} \Pr\{\xi_N(x, t+h) = k | \xi_N(x, t) = k', \mathbf{p}(t)\} \Pr\{\xi(x, t) = k'\} = \\ &= \Pr\{\xi_N(x, t+h) = k, k > 0 | \xi_N(x, t) = k, \mathbf{p}(t)\} p_k(t) + \\ &+ \Pr\{\xi_N(x, t+h) = k, k > 0 | \xi_N(x, t) = 0, \mathbf{p}(t)\} \left[1 - \sum_{i=1}^K p_i(t) \right] = \\ &= [1 - M_k(\mathbf{p}(t))h] p_k(t) + \Lambda_k(\mathbf{p}(t)) \left[1 - \sum_{i=1}^K p_i(t) \right] h + o(h) = \\ &= p_k(t) + \left\{ \left[1 - \sum_{i=1}^K p_i(t) \right] \Lambda_k(\mathbf{p}(t)) - p_k(t) M_k(\mathbf{p}(t)) \right\} h + o(h). \end{aligned}$$

The result follows from

$$\frac{dp_k(t)}{dt} = \lim_{h \rightarrow 0} \frac{p_k(t+h) - p_k(t)}{h}.$$

□

2. APPEARANCE AND HEALING OF MICROCRACKS

We imagine an infinite solid volume to be subdivided into equal cubic cells with edges perpendicular to coordinate axes x_1, x_2, x_3 and with centers at the points of a cubic lattice \mathbb{Z}_a^3 , a is the step of the lattice. We prescribe at boundaries of the body a uniform symmetric tensor of second order (stress tensor) $\sigma \equiv \sigma_{kl}(t)$, $k, l = 1, 2, 3$, that is a function of, in general, time t . A

configuration ζ_t has at $x \in \mathbb{Z}_a^3$ the value $\zeta_{x,t} = u_k$, $k = 1, \dots, K$. if the cell centered at x contains a planar disk (microcrack) centered at the same point, with radius $r < a/2$ and with normal u_k . The value $\zeta_{x,t} = u_0$ corresponds to an empty cell.

We define a multicomponent mean field birth-death process of microcracks appearance and disappearance by equations

$$\frac{dp_k(t)}{dt} = \left[1 - \sum_{i=1}^K p_i(t) \right] \Lambda_k - p_k(t) M_k \quad (2.1)$$

with the intensities as defined below.

The intensity $M_k \equiv M_k(\mathbf{p}(t))$ of microcrack healing has the form

$$M_k = c_{0k} \exp \{ -\beta U_k(\sigma; \mathbf{p}(t)) \} \quad (2.2)$$

where c_{0k} are the numeric constants, $\beta = \frac{1}{k_B T}$, T is temperature, k_B is Boltzmann's constant, σ is the stress tensor, $U_k(\sigma; \mathbf{p}(t))$ are activation energies of the healing. We use here the random truncation of a continuous healing process. If the component of the stress tensor normal to the microcrack plane is compressive, the crack is closed. During the healing of the closed crack its opposite sides stick together and their relative displacement is remembered and carries this contribution to the residual strain. If the component of the stress tensor normal to the microcrack plane is tensile, the crack is open. Its healing is the filling of the cavity with molecules from the host material. The strain brought about by the relative displacement of crack sides is remembered but the volume part of the strain disappears, because the density of the material remains unchanged.

We suppose that the rate of healing is proportional to the rate of diffusion. The diffusion coefficient D is specified by the Hevesy formula (see, e.g., (author?) [Frenkel 1946])

$$D = D_0 \exp \{ -\beta E_a \},$$

where D_0 is constant and E_a is activation energy. In our case activation energies $U_k(\sigma; \mathbf{p}(t))$ depend in general on the stress normal to the microcrack plane and are different for open and closed cracks. It is convenient to assume that a crack exists without change until the moment of healing, at which it disappears together with its contribution to the stress field.

To define the intensity $\Lambda_k \equiv \Lambda_k(\mathbf{p}(t))$ of microcrack birth let's assume that any cell contains K types of microdefects with molecular size. A defect of each type can lose its stability and become a microcrack of the same type.

The intensity λ_k has the form

$$\Lambda_k = c_{1k} \exp \{ -\beta [H - E_k^0(\sigma; \mathbf{p}(t))] \} \quad (2.3)$$

where c_{1k} are constants, H is activation energy (that is, the thermofluctuative elastic energy threshold where a microdefect loses stability and becomes a mi-

crocrack), $E_k^0(\sigma; \mathbf{p}(t))$ is the additional elastic energy brought in by a k -type microdefect.

3. MEAN STRESS FIELD

The stress field in a body with an arbitrary microcrack system cannot be represented explicitly. Therefore we use an approximation; roughly speaking, we assume that a reduction of stress in the neighborhoods of cracks is compensated by an increase of stress outside of these neighborhoods, the mean stress over the volume being kept equal to its value at the boundary. (A similar approach is used in problems arising in breaking of ropes composed of many wires and also in strength models for solid bodies under axial tension.)

We assume that the stress tensor outside of spheres of radii r circumscribed around microcracks is uniform and is specified by an effective tensor $\bar{\sigma}$. Inside of these spheres the stress is also uniform and differs from $\bar{\sigma}$ by some zero components when they vanish at the crack surface. It is clear that we only approximate a continuous stress field by discontinuous functions and by no means assume that stress undergoes actual discontinuities at the surface of the spheres.

Let us choose coordinates x_1^i , x_2^i , and x_3^i for i -type microcracks in such a way that the x_1^i axis is perpendicular to microcrack planes. Denote by $\mathbf{A}_{kl}^{(i)}$ the matrix elements specifying the transformation $\mathbf{A}^{(i)}$ of coordinates x_1, x_2, x_3 to x_1^i, x_2^i, x_3^i . Let $\mathbf{T}^{(i)}\bar{\sigma} : (\mathbf{T}^{(i)}\bar{\sigma})_{kl} = \mathbf{A}_{km}^{(i)}\mathbf{A}_{ln}^{(i)}\bar{\sigma}_{mn}$ be the tensor $\bar{\sigma}$ in the new coordinates (a notational convenience introduced by Einstein will be used here: tensor sums are taken over all repeated subscripts). We introduce the piecewise linear operator \mathbf{S} by the rule: $(Sa)_{kl} = S_{kl}a_{kl}$ (in this unique case the summation over repeated subscripts isn't made) for any symmetric tensor a , where $\mathbf{S} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$ if $a_{11} \geq 0$, $\mathbf{S} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$ if $a_{11} < 0$. The case $a_{11} \geq 0$ corresponds to the tensile normal stress applied on the plane of an open crack. The case $a_{11} < 0$ corresponds to the compressive normal stress applied on the plane of a closed crack. Let us define an operator $\mathbf{Q}^{(i)}$ which is applied to $\bar{\sigma}$ by

$$\mathbf{Q}^{(i)}\bar{\sigma} = \mathbf{T}^{(i)-1}\mathbf{S}\mathbf{T}^{(i)}\bar{\sigma}.$$

Piecewise linear operators $\mathbf{Q}^{(i)}$ specifies stress in the neighborhood of cracks and remove those stress tensor components that are tangent to the crack plane and normal to the crack plane component if it is positive, i.e. tensile (it is the case of an "open" crack).

We assume that $\theta = \frac{4\pi r^3}{3a^3}$, then $\theta p_i(t)$ is the relative volume with stress $\mathbf{Q}^{(i)}\bar{\sigma}$, and the tensor $\bar{\sigma}$ is the solution of the system of equations

$$\left[\mathbf{I} - \sum_{i=1}^K \theta p_i(t) (\mathbf{I} - \mathbf{Q}^{(i)}) \right] \bar{\sigma} = \sigma. \quad (3.1)$$

4. ELASTIC ENERGY

The density of elastic energy $e(\sigma)$ in a homogeneous body under stress $\sigma \equiv \sigma_{ij}$ is given by

$$e(s) = \frac{(\sigma:\varepsilon)}{2} = \frac{(\sigma:\mu\sigma)}{2} = \frac{(\lambda\varepsilon:\varepsilon)}{2}$$

((author?)) [Landau 1986]), where $\varepsilon \equiv \varepsilon_{kl}$ is the elastic strain tensor, the tensor of the fourth order $\lambda \equiv \lambda_{ijmn}$ is the stiffness tensor, $\mu \equiv \mu_{ijmn}$ is inverse for λ elastic pliability tensor, $\sigma = \lambda\varepsilon$ ($\sigma_{ij} = \lambda_{ijmn}\varepsilon_{mn}$) is generalized Hooke's law, $(\sigma:\varepsilon) \equiv \sigma_{ij}\varepsilon_{ij}$ is double inner product of tensors. (We use in this paper the approximation of small deformations, where Hooke's law is fulfilled at all values of the strain tensor.)

In a homogeneous isotropic body under the stress σ the density of elastic energy $\tilde{e}(\sigma)$ has the form

$$\tilde{e}(\sigma) = \frac{1}{2E}(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2) + \frac{1+\nu}{E}(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2)$$

where E is Young's modulus and ν is Poisson's ratio. Let A be a body constructed from a cubic set of cells in \mathbb{Z}_a^3 , n_k be the number of cells with k -type cracks, $\mathbf{p}^{(A)} = (p_1^{(A)}, \dots, p_K^{(A)})$, $p_k^{(A)} = \frac{n_k}{|A|}$, $\rho_k^{(A)} = \theta p_k^{(A)}$. It is easy to see that the elastic energy $E_A(\sigma; \mathbf{p}^{(A)})$ of the body with cracks under the stress σ is given by

$$\begin{aligned} E_A(\sigma; \mathbf{p}^{(A)}) &= a^3 |A| e_A(\sigma; \mathbf{p}^{(A)}) = \\ &= a^3 |A| \left\{ \left[1 - \sum_{k=1}^K \rho_k^{(A)} \right] \tilde{e}(\bar{\sigma}) + \sum_{k=1}^K \rho_k^{(A)} \tilde{e}(\mathbf{Q}^{(k)} \bar{\sigma}) \right\} = \\ &= a^3 |A| \left\{ \left[1 - \sum_{k=1}^K \rho_k^{(A)} \right] \bar{\sigma}_{ij} \mu_{ijkl}^0 \bar{\sigma}_{kl} + \sum_{k=1}^K \rho_k^{(A)} (\mathbf{Q}^{(k)} \bar{\sigma})_{ij} \mu_{ijkl}^0 (\mathbf{Q}^{(k)} \bar{\sigma})_{kl} \right\} \end{aligned}$$

and the density of elastic energy for the infinite body

$$\begin{aligned} e &\equiv e(\sigma; \mathbf{p}) = \\ &= \lim_{|A| \rightarrow \infty} e_A(\sigma; \mathbf{p}^{(A)}) = \left[1 - \sum_{k=1}^K \rho_k^{(A)} \right] \tilde{e}(\bar{\sigma}) + \sum_{k=1}^K \rho_k^{(A)} \tilde{e}(\mathbf{Q}^{(k)} \bar{\sigma}) \end{aligned}$$

The elastic energy $E_{k,A}(\sigma; \mathbf{p}^{(A)})$ added by a k -type crack has the form

$$\begin{aligned} E_{k,A}(\sigma; \mathbf{p}^{(A)}) &= a^3 |A| \left[e_A(\sigma; p_1^{(A)}, \dots, p_k^{(A)} + \frac{1}{|A|}, \dots, p_K^{(A)}) - e_A(\sigma; \mathbf{p}^{(A)}) \right] = \\ &= a^3 \frac{\partial e_A(\sigma; \mathbf{p}^{(A)})}{\partial p_k^{(A)}} + O(|A|^{-1}) \end{aligned}$$

and for the infinite lattice

$$E_k(\sigma; \mathbf{p}) = a^3 e_k = a^3 \frac{\partial e}{\partial p_k},$$

where $e_i \equiv e_i(\sigma; p_t(j), j = 1, \dots, I)$ is the density of elastic energy in the cell with a i -type crack. To express intensity (2.3) explicitly we define the energy of microdefects generating cracks in a similar form

$$E_i^0(\sigma; \mathbf{p}) = a_0^3 e_i$$

where a_0 is of the order of intermolecular distance.

The density of elastic energy can be expressed in the form

$$e = \frac{1}{2} \mu_{ijmn}(\sigma; \mathbf{p}) \sigma_{ij} \sigma_{mn} \equiv \frac{1}{2} (\sigma : \mu \sigma), \quad (4.1)$$

where μ is the effective elastic pliability tensor, $\mu_{ijmn} \equiv \mu_{ijmn}(\sigma; \mathbf{p})$. As the component s_{11} of the operator \mathbf{S} has the jump when the normal stress applied on the plane of the crack changes the sign, the dependence $\bar{\sigma}$ on σ is piecewise linear and μ_{ijmn} are step functions of σ .

Similarly,

$$e_k = \frac{1}{2} \mu_{klmn}^k \sigma_{kl} \sigma_{mn} \equiv \frac{1}{2} (\sigma : \mu^k \sigma), \quad (4.2)$$

where tensor components $\mu_{ijmn}^k \equiv \mu_{ijmn}^k(\sigma; \mathbf{p})$ are step functions of σ .

5. EQUATIONS OF NON-EQUILIBRIUM THERMODYNAMICS

Consider deformation in a body under stress. A tensor u_{ij} of the total macroscopic strain consists of the tensor of reversible (it vanishes if the stress is zero) elastic strain and the tensor of irreversible residual strain $r_{ij} = u_{ij} - \varepsilon_{ij}$ appearing when microcracks are healing.

The total macroscopic strain was defined previously (**author?**) [Gertzik 1998] by

$$u_{ij} = \lim_{n \rightarrow \infty} \frac{1}{|A_n| a^3} \int_S \frac{1}{2} (X_i n_j + X_j n_i) ds$$

where A_n are cubes with centers at origin, $A_n \subseteq A_{n+1}$, $\cup_{n=1}^{\infty} A_n = \mathbb{Z}^{\nu}$, X_i are displacement components, n_i are normal components of the surface, and the integral is taken over the surface S of the cube. It has been shown (see (4.1)) that macroscopic elastic strain ε_{ij} has the form

$$\varepsilon_{ij} = \frac{\partial e}{\partial \sigma_{ij}} = \mu_{ijmn} \sigma_{mn} \quad (5.1)$$

if the crack configuration is fixed.

During the time interval dt the density of k -type crack increases by $\left[1 - \sum_{l=1}^K p_l\right] \Lambda_k dt$ and the strain increment $du_{ij}^{k(+)}$, in accordance with (4.2), has the form

$$du_{ij}^{k(+)} = \mu_{ijmn}^k \sigma_{mn} \left[1 - \sum_{l=1}^K p_l(t)\right] \Lambda_k dt \quad (5.2)$$

We remind that the stress tensor σ_{ij} can be expressed as the sum of two other stress tensors: the mean hydrostatic stress tensor $\bar{\sigma} \delta_{ij}$ which tends to change the volume of the stressed body, and the deviatoric component called the stress deviator tensor, s_{ij} which tends to distort it: $\sigma_{ij} = s_{ij} + \bar{\sigma} \delta_{ij}$ where $\bar{\sigma}$ is the mean stress given by $\bar{\sigma} = \frac{\sigma_{ii}}{3}$. After some crack has been healed, no work is done by the stress deviator and the strain deviator does not change (the displaced sides of a closed crack stick together and the strain caused by it becomes residual instead of elastic strain) but every diagonal element of the strain tensor u_{ij} decreases by 1/3 of volume strain due to the crack (as the density of the material remains constant, the volume strain due to the open crack disappears), that is, the strain decrement $du_{ij}^{k(-)}$, in accordance with (4.2) is of the form

$$du_{ij}^{k(-)} = \tilde{\mu}_{ijmn}^k \sigma_{mn} p_k(t) M_k dt$$

where

$$\tilde{\mu}_{ijmn}^k = \begin{cases} \delta_{ij} \mu_{ijmn}^k / 3, & \text{if crack is open,} \\ 0, & \text{if crack is closed.} \end{cases}$$

The increment $du_{ij}|_{\sigma}$ of the strain ε_{ij} during time interval dt for a constant σ is represented by

$$\begin{aligned} du_{ij}|_{\sigma} &= \sum_{k=1}^K [du_{ij}^{k(+)} - du_{ij}^{k(-)}] = \\ &= \sum_{k=1}^K \left(\mu_{ijmn}^k \sigma_{mn} \left[1 - \sum_{l=1}^K p_l(t)\right] \Lambda_k - \tilde{\mu}_{ijmn}^k \sigma_{mn} p_k(t) M_k \right) dt \end{aligned}$$

The last expression and (5.1) yield the total strain increment

$$du = \mu d\sigma + \sum_{k=1}^K \left(\left[1 - \sum_{l=1}^K p_l(t)\right] \Lambda_k \mu^k - p_k(t) M_k \tilde{\mu}^k \right) \sigma dt \quad (5.3)$$

which shows that appearance and disappearance of cracks give rise to viscous strain component.

Using (2.1) and the relation

$$\begin{aligned} d\mu_{ijmn} &= d\left(\frac{\partial e}{\partial \sigma_{ij} \partial \sigma_{mn}}\right) = \frac{\partial^2}{\partial \sigma_{ij} \partial \sigma_{mn}} \sum_{k=1}^K \frac{\partial e}{\partial p_k} dp_k = \\ &= \sum_{k=1}^K \frac{\partial^2}{\partial \sigma_{ij} \partial \sigma_{mn}} e_k dp_k = \sum_{k=1}^K \mu_{ijmn}^k dp_k, \end{aligned}$$

i.e.

$$d\mu = \sum_{k=1}^K \mu^k \left(\left[1 - \sum_{l=1}^K p_l(t) \right] \Lambda_k - p_k(t) M_k \right) dt \quad (5.4)$$

we have

$$d\varepsilon = \mu d\sigma + \sum_{k=1}^K \mu^k \left(\left[1 - \sum_{l=1}^K p_l(t) \right] \Lambda_k - p_k(t) M_k \right) \sigma dt$$

and

$$dr = \sum_{k=1}^K (\mu^k - \tilde{\mu}^k) p_k(t) M_k \sigma dt.$$

When some procedure of strain change is chosen as a prescribed external condition, then the last formula leads to the equation for external stress

$$d\sigma = \lambda \left\{ du - \sum_{k=1}^K \left(\left[1 - \sum_{l=1}^K p_l(t) \right] \Lambda_k \mu^k - p_k(t) M_k \tilde{\mu}^k \right) \sigma dt \right\} \quad (5.5)$$

where λ is the inverse of μ the effective stiffness tensor (in general, anisotropic) for a body with cracks.

The crack surface energy is represented by $\pi r^2 \gamma$, where γ is the density of surface energy. The density of internal energy for the body (apart from an additive constant independent of external conditions and crack densities) is the sum of the density of elastic energy, surface crack energy, and thermal (vibrational) energy of atoms. The first law of thermodynamics can be written in that case, in view of (4.1), as

$$\begin{aligned} \frac{1}{2} d(\sigma \cdot \mu \sigma) + \gamma \frac{\pi r^2}{a^3} \sum_{k=1}^K \left(\left[1 - \sum_{l=1}^K p_l(t) \right] \Lambda_k - p_k(t) M_k \right) dt + \rho C_p dT = \\ = (\sigma \cdot du) + dQ \end{aligned}$$

where ρ is the density of the material, C_p is specific heat at constant pressure, and dQ is the specific heat energy increment from the outside. Using (5.3, 2.1, 5.4)

and the equality $\mu_{ijmn} = \mu_{mnij}$ we get the expression for the temperature increment:

$$\begin{aligned} \rho C_p dT = dQ + \sum_{k=1}^K \left[\frac{1}{2} (\sigma \cdot \mu^k \sigma) - \gamma \frac{\pi r_i^2}{a^3} \right] \left[1 - \sum_{l=1}^K p_l(t) \right] \Lambda_k dt + \\ + \sum_{k=1}^K \left[\frac{1}{2} (\sigma \cdot \mu^k \sigma) + \gamma \frac{\pi r_i^2}{a^3} - (\sigma \cdot \tilde{\mu}^k \sigma) \right] p_k(t) M_k dt \end{aligned} \quad (5.6)$$

Equations (??), (2.1), (5.5) and (5.6) with (2.2), (2.3) constitute a closed system describing the non-equilibrium thermodynamics of a deformation process in a solid body (as usual, superior dots denote differentiation with respect to time):

$$\frac{dp_k(t)}{dt} = \left[1 - \sum_{i=1}^K p_i(t) \right] \Lambda_k - p_k(t) M_k, \quad k = 1, \dots, K \quad (5.7)$$

$$\dot{\sigma} = \lambda \left\{ \dot{u} - \sum_{k=1}^K \left(\left[1 - \sum_{l=1}^K p_l(t) \right] \Lambda_k \mu^k - p_k(t) M_k \tilde{\mu}^k \right) \sigma \right\} \quad (5.8)$$

$$\begin{aligned} \rho C_p \dot{T} = \left[1 - \sum_{l=1}^K p_l(t) \right] \Lambda_k \sum_{k=1}^K \left[\frac{1}{2} (\sigma \cdot \mu^k \sigma) - \gamma \frac{\pi r_i^2}{a^3} \right] + \\ + \sum_{k=1}^K \left[\frac{1}{2} (\sigma \cdot \mu^k \sigma) + \gamma \frac{\pi r_i^2}{a^3} - (\sigma \cdot \tilde{\mu}^k \sigma) \right] p_k(t) M_k + G \end{aligned} \quad (5.9)$$

where G is the heat influx rate per unit volume. It is necessary to notice that any nonzero stress does a work that dissipates irreversibly and the system is not equilibrium even in the stationary case.

6. POWER OF ACOUSTIC EMISSION

According to (4.2), (5.2) the external forces σ make at occurrence of cracks the differential work

$$\begin{aligned} \sum_{k=1}^K \sigma_{ij} du_{ij}^{k(+)} &= \left[1 - \sum_{k=1}^K p_k(t) \right] \sum_{k=1}^K \sigma_{ij} \mu_{ijmn}^k \sigma_{mn} \Lambda_k dt = \\ &= \left[1 - \sum_{k=1}^K p_k(t) \right] \sum_{k=1}^K (\sigma : \mu^k \sigma) \Lambda_k dt. \end{aligned}$$

and the increment of density of elastic energy due to new microcracks is equal to $\frac{1}{2} \left[1 - \sum_{k=1}^K p_k(t) \right] \sum_{k=1}^K (\sigma : \mu^k \sigma) \Lambda_k dt$. Thus only half of the work goes elastic energy and the second half dissipates through acoustic emission and subsequently becomes heat. This is caused by spasmodic changes in the effective elastic moduli of the cells where cracks appear. Therefore the power of acoustic emission per unit volume w is

$$\mathbf{w} = \frac{1}{2} \left[\mathbf{1} - \sum_{k=1}^K p_k(t) \right] \sum_{k=1}^K (\boldsymbol{\sigma} : \boldsymbol{\mu}^k \boldsymbol{\sigma}) \boldsymbol{\Lambda}_k. \quad (6.1)$$

7. DYNAMICS OF A CONTINUUM

To derive dynamic equations for continuum from equations for a material point we introduce velocities $\mathbf{v} \equiv v_k(\mathbf{x}, t)$, $k = 1, 2, 3$ in three-dimensional space, $\mathbf{x} = (x_1, x_2, x_3)$. Euler vector coordinates $\mathbf{x} \equiv \mathbf{x}(\mathbf{X}, t)$ are functions of Lagrange (material) coordinates \mathbf{X} of a continuum point in some initial configuration (see, e.g., **(author?)** [Day 1972]) and the velocity of the point is $\mathbf{v} \equiv \mathbf{v}(\mathbf{X}, t) = \frac{\partial \mathbf{x}}{\partial t}$. Let f be a local parameter of the continuum. This can be considered to be both $f(\mathbf{X}, t)$ and $f(\mathbf{x}, t)$.

The full derivative of f with respect to time $\dot{f} \equiv \partial f(\mathbf{X}, t)/\partial t$ is equal to

$$\dot{f} \equiv \partial f(\mathbf{x}(\mathbf{X}, t), t)/\partial t + \sum_{i=1}^3 \frac{\partial f(\mathbf{x}, t)}{\partial x_i} \frac{\partial x_i}{\partial t} = \frac{\partial f(\mathbf{x}, t)}{\partial t} + (\mathbf{v} \cdot \nabla f(\mathbf{x}, t))$$

for any differentiable function f .

To calculate the strain rate we use the small strain approximation (the finite strain theory is too bulky to develop it here). In this approximation the Lagrange and Euler coordinates are identical and the small displacement vector $U_i(x, t)$, $i = 1, 2, 3$, is defined at each point. Symmetrized velocity gradient $\nabla^{(s)} \mathbf{v} = \frac{1}{2} (\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k})$ is by definition $u_{lk} = \frac{1}{2} \left(\frac{\partial U_l}{\partial x_k} + \frac{\partial U_k}{\partial x_l} \right)$ equal to the strain rate tensor $\nabla^{(s)} \mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}$.

The law of conservation of momentum (Newton's second law) takes the form

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\rho(\mathbf{v} \cdot \nabla \mathbf{v}) + \text{div } \boldsymbol{\sigma} + \rho \mathbf{F}$$

where ρ is the density and \mathbf{F} is the force per unit mass. Using (5.7) we get a kinetic equation for crack densities

$$\frac{\partial p_k}{\partial t} = -(\mathbf{v} \cdot \nabla p_k) + \left[\mathbf{1} - \sum_{i=1}^K p_i(t) \right] \boldsymbol{\Lambda}_k - p_k \mathbf{M}_k, \quad k = 1, \dots, K$$

Equations (5.8) lead to equations of state for the continuum:

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} = -(\mathbf{v} \cdot \nabla \boldsymbol{\sigma}) + \lambda \left\{ \nabla^{(s)} \mathbf{v} - \sum_{k=1}^K \left(\left[\mathbf{1} - \sum_{l=1}^K p_l \right] \boldsymbol{\Lambda}_k \boldsymbol{\mu}^k - p_k \mathbf{M}_k \tilde{\boldsymbol{\mu}}^k \right) \right\}$$

Lastly, we get the equation of heat conduction with heat sources using the law of conservation of energy in the form (5.9) and the Fourier law $\mathbf{q} = \kappa \nabla T$ where

q is the heat flux vector, $G = \text{div } q$, κ is thermal conductivity:

$$\begin{aligned} \frac{\partial T}{\partial t} = & -(\mathbf{v} \cdot \nabla T) + (\rho C_p)^{-1} \left\{ \left[1 - \sum_{l=1}^K p_l(t) \right] \Lambda_k \sum_{k=1}^K \left[\frac{1}{2} (\sigma \cdot \mu^k \sigma) - \gamma \frac{\pi r_i^2}{a^3} \right] + \right. \\ & \left. + \sum_{k=1}^K \left[\frac{1}{2} (\sigma \cdot \mu^k \sigma) + \gamma \frac{\pi r_i^2}{a^3} - (\sigma \cdot \tilde{\mu}^k \sigma) \right] p_k(t) M_k \right\} + \chi \Delta T, \end{aligned}$$

$\chi = \kappa(\rho C_p)^{-1}$ is thermal diffusivity.

8. PHASE TRANSITION

Let $\sigma_{12} = \sigma_{21} \geq 0$ be external shear stress applied to the solid, $\sigma = \sigma_{12}/\sqrt{\mu_s}$, μ_s is a the shear modulus, $U(\sigma; \mathbf{p}(t)) = U$, U is a constant, and allowed cracks have their normal parallel to axes x_1 and x_2 . Then (2.2) - (3.1) give

$$\begin{aligned} \bar{\sigma} &= \frac{\sigma}{1 - \theta p}, \\ \Lambda_1 = \Lambda_2 = \Lambda &= c_1 \exp \left\{ \beta \left[a_0 \left(\frac{\sigma}{1 - \theta p} \right)^2 - H \right] \right\}, \\ M_1 = M_2 = M &= c_0 \exp \{-\beta U\}, \end{aligned}$$

and the two types of cracks are indistinguishable in (5.7). We introduce their total density p , and get from (5.7) the equation for stationary case

$$c_1 (1 - p) \exp \left\{ \beta \left[a_0 \left(\frac{\sigma}{1 - \theta p} \right)^2 - H \right] \right\} - c_0 p \exp \{-\beta U\} = 0.$$

From this we get the dependence of σ on p and β

$$\sigma = \sqrt{\frac{1}{a_0 \beta} (1 - \theta p)^2 \left[\ln \frac{p}{1 - p} - \ln \frac{c_1}{c_0} + \beta (H - U) \right]} \quad (8.1)$$

and $\sigma \geq 0$ when

$$\frac{c_1 \exp \{-\beta H\}}{c_1 \exp \{-\beta H\} + c_0 \exp \{-\beta U\}} \leq p < 1.$$

As σ changes from 0 to ∞ , the phase transition takes place if $\min \frac{d\sigma}{dp} \leq 0$. At $\min \frac{d\sigma}{dp} < 0$ the dependence of p on σ has an S -shaped form (see Fig.1) similar to the van der Waals curve.

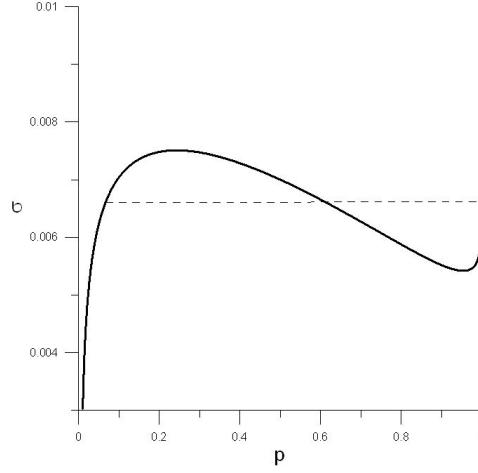


Fig. 1

Following the technique of pioneer work of M.Kac for the Curie-Weiss model (**author?**) [Kac 1968] we construct from (A.3), (A.4) (see Appendix) the asymptotic distribution $u_N(x)$ of $x = \frac{n}{N}$ at large N :

$$\begin{aligned}
 u_N(x) &\equiv \Pr\{n = xN\} = \sum_{A: |A|=xN} P_N(A) \sim \\
 &\sim \frac{N!}{(N-xN)!(xN)!} \frac{1}{Z_N \sqrt{\Lambda(y)M(y)}} \exp \left\{ N \int_0^x \ln f(z) dz \right\} \sim \\
 &\sim \frac{1}{Z_N} \sqrt{\frac{1}{2\pi N(1-x)x\Lambda(y)M(y)}} \exp \{NF(x)\}, \quad (8.2)
 \end{aligned}$$

$$F(x) = x \ln \frac{c_1}{c_0} + x\beta \left(U - H + a_0 \frac{\sigma^2}{1 - \theta x} \right) - x \ln x - (1-x) \ln(1-x),$$

$$Z_N \sim \int_0^1 \sqrt{\frac{1}{2\pi N(1-x)x\Lambda(y)M(y)}} \exp \{NF(x)\} dx.$$

As $N \rightarrow \infty$ the probability measure tends to a measure concentrated at a points of a strict maximum of $F(x)$ and a first-order phase transition takes place if $F(x)$ has two identical maxima. The condition of extremum of $F(x)$

$$\frac{dF}{dx} = \ln \frac{c_1}{c_0} + \beta (U - H) + a_0 \beta \frac{\sigma^2}{(1 - \theta x)^2} - \ln \frac{x}{1-x} = 0$$

is equivalent to (8.1). The requirement of equality of two maxima $F(x_1) = F(x_2)$ of function $F(x)$ yields an analog of Maxwell equal area rule

$$\int_{x_1}^{x_2} \left[\ln \frac{c_1}{c_0} + \beta_c (U - H) + a_0 \beta \frac{\sigma_c^2}{(1 - \theta z)^2} - \ln \frac{z}{1 - z} \right] dz = 0.$$

A second-order phase transition takes place at critical point (p_c, β_c, σ_c) where $\frac{dF}{dx}|_{p_c} = \frac{d^2F}{dx^2}|_{p_c} = \frac{d^3F}{dx^3}|_{p_c} = 0$. From

$$\frac{dF}{dx} |_{p_c} = \ln \frac{c_1}{c_0} + \beta_c (U - H) + a_0 \beta \frac{\sigma_c^2}{(1 - \theta p_c)^2} - \ln \frac{p_c}{1 - p_c} = 0,$$

$$\frac{d^2F}{dx^2} |_{p_c} = 2a_0 \beta_c \frac{\theta \sigma_c^2}{(1 - \theta p_c)^3} - \frac{1}{p_c (1 - p_c)} = 0,$$

$$\frac{d^3F}{dx^3} |_{p_c} = 6a_0 \beta_c \frac{\theta^2 \sigma_c^2}{(1 - \theta p_c)^4} + \frac{1 - 2p_c}{p_c^2 (1 - p_c)^2} = \frac{1 - 2(1 - \theta)p_c - \theta p_c^2}{(1 - \theta p_c)p_c^2 (1 - p_c)^2} = 0,$$

we find coordinates of the critical point at $H \neq W$

$$p_c = \frac{\sqrt{1 - \theta + \theta^2} - 1 + \theta}{\theta},$$

$$\beta_c = \frac{1}{H - U} \left[\ln \frac{c_1}{c_0} + \frac{(2 - \theta - \sqrt{1 - \theta + \theta^2}) \theta}{(2 - \theta)\sqrt{1 - \theta + \theta^2} - 2(1 - \theta) - \theta^2} - \ln \frac{\sqrt{1 - \theta + \theta^2} - 1 + \theta}{1 - \sqrt{1 - \theta + \theta^2}} \right],$$

$$\sigma_c = \sqrt{\frac{1}{a_0 \beta_c} \frac{(1 - \theta + \theta^2) (2 - \theta - \sqrt{1 - \theta + \theta^2}) \theta}{(2 - \theta)\sqrt{1 - \theta + \theta^2} - 2(1 - \theta) - \theta^2}}.$$

We derive the same results from (8.1) with $(p, \beta, \sigma) = (p_c, \beta_c, \sigma_c)$ and $\frac{d\sigma}{dx}|_{p_c} = \frac{d^2\sigma}{dx^2}|_{p_c} = 0$.

In the case $H = U$ we have

$$p_c = \frac{\sqrt{1 - \theta + \theta^2} - 1 + \theta}{\theta},$$

$$\beta_c \sigma_c^2 = \frac{(1 - \theta p_c)^2}{a_0} \left(\ln \frac{p_c}{1 - p_c} - \ln \frac{c_1}{c_0} \right).$$

In this case there is the line of critical points $\beta_c \sigma_c^2 = \text{const}$ on the plane (β, σ) instead of a single critical point in the ordinary case.

9. DISTRIBUTION OF LOGARITHMIC POWER OF ACOUSTIC EMISSION AT THE CRITICAL POINT FOR A LARGE SYSTEM

In the conditions of the previous section the power of acoustic emission W for a solid with N cells has the form (6.1)

$$W = \frac{N}{2}(1-p) \frac{\theta\sigma^2}{(1-\theta p)^2} c_1 \exp \left\{ \beta \left[a_0 \left(\frac{\sigma}{1-\theta p} \right)^2 - H \right] \right\}.$$

To find the distribution $\varphi_N(y)$ of $y = \ln W$ at the critical point we notice that the distribution $u_N(x)$ of crack density $x = \frac{n}{N}$ at large N , according to (8.2), looks at the critical point like

$$\begin{aligned} u_N(x) &\sim \frac{1}{Z_N} \sqrt{\frac{1}{2\pi N(1-x)x\Lambda(y)M(y)}} \exp \{NF(x)\} \sim \\ &\sim u_N(p_c) \exp \left\{ \frac{N}{(2n)!} \frac{d^{2n}F_c}{dx^{2n}} \Big|_{p_c} (x-p_c)^{2n} - \frac{1}{2} \frac{d}{dx} \ln [\Lambda(x)M(x)(1-x)x] \Big|_{p_c} (x-p_c) \right\}, \end{aligned}$$

at the critical point where n is the smallest positive number at which the derivative $\frac{d^{2n}F_c}{dx^{2n}}$ is distinct from 0 ($\frac{d^{2n-1}F_c}{dx^{2n-1}} \Big|_{p_c} = 0$ because $u_N(x)$ has a maximum at the point p_c). As

$$\frac{d^4F_c}{dx^4} \Big|_{p_c} = \frac{-2(1-\theta+\theta p_c)}{(1-\theta p_c)p_c^2(1-p_c)^2} \leq 0,$$

we have

$$u_N(x) \sim \frac{1}{Z_N} \exp \{NF_c(p_c)\} \exp \left\{ \frac{N}{4!} \frac{d^4F_c}{dx^4} \Big|_{p_c} (x-p_c)^4 - \frac{1}{2} \frac{d}{dx} [\ln \Lambda(x)M(x)(1-x)x] \Big|_{p_c} (x-p_c) \right\}.$$

The distribution $\varphi_N(y)$ is expressed by $u_N(x)$ as

$$\varphi_N(y) = \sum_{x: \ln W(x)=y} u(x(y)) \left(\frac{dy}{dx} \right)^{-1} = \sum_{x: W(x)=y} u(x(y)) W \left(\frac{dW}{dx} \right)^{-1},$$

and

$$\frac{dW}{dx} = W \left[\frac{2\theta}{(1-\theta x)} - \frac{1}{1-x} + \frac{2\alpha\beta\sigma^2\theta}{(1-\theta x)^3} \right] \equiv Wg(x).$$

Therefore $\varphi_N(y) = \sum_{x: W(x)=y} \frac{u(x(y))}{g(x)}$ and at the critical point

$$\varphi_N(y) = \sum_{x:W(x)=y} \frac{u(x(y))}{g(x)} \sim \left\{ C \exp \frac{N}{4!} \frac{d^4 F_c}{dx^4} \Big|_{p_c} \frac{1}{g(p_c)^4} (y - \ln W_c)^4 - \right. \\ \left. - \frac{1}{2} \frac{d}{dx} \ln [g(x)\Lambda(x)M(x)(1-x)x] \Big|_{p_c} \frac{1}{g(p_c)} (y - \ln W_c) \right\},$$

As the energy ΔE emitted during a small time Δt is $W \Delta t$, the logarithm of its distribution has the same form as $\ln \varphi_N$

$$\ln \varphi_N = a - b \ln W - c (\ln W - \ln W_c)^4. \quad (9.1)$$

This expression is surprisingly similar to the empirical distribution of earthquakes energy. Theoretically it satisfies to the Gutenberg-Richter law (**author?**) [Gutenberg 1954] $\ln \frac{N_E}{N_{total}} = a - b \ln E$, N_E is the number of earthquakes with energy E , N_{total} is total number of earthquakes, but empirical curves deviate from linear law downwards at both edges just as in (9.1). Fig. 2 shows example of a plot of φ_N , and on Fig. 3 we see the top part of the plot of $\ln \varphi_N$ and corresponding linear dependence.

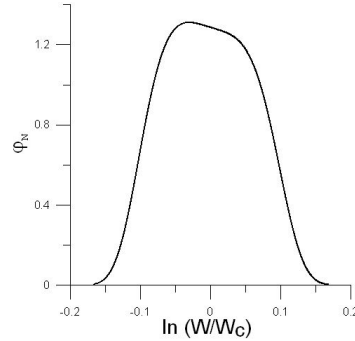


Fig. 2

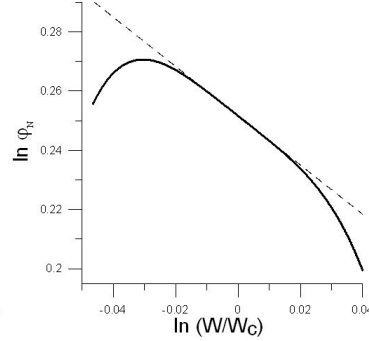


Fig.3

If we assume that the area of the stress reduction is not less than the sphere inscribed in a cubic cell, then $\frac{\pi}{6} \leq \theta < 1$, and $0.271 \leq b < 0.5$. The Gutenberg-Richter law expresses the relationship between magnitude M and the number of earthquakes with this M . The magnitude is expressed through energy in joules ((**author?**) [Kasahara 1981]) $M = \frac{2}{3}(\lg E - 11.8)$. The relationship between $\frac{2}{3} \lg W$ and $\lg \varphi_N$ has the form $\lg \varphi_N = a' - b' \lg W - c' (\lg W - \lg W_c)^4$ with $0.407 \leq b' < 0.75$. This range of b' is comparable with the range $0.5 \leq b' < 1.5$ observed for the distribution of earthquake magnitude.

10. ZHURKOV'S CURVES

In (author?) [Gertzik 1994] it was demonstrated that the numerical solutions of equations (5.7)-(5.9) reproduce a number of physical properties of solids observed in experiments. These properties include creep, the existence of lower and upper yield points, strain hardening, dilatancy, the growth of elastic anisotropy and drop in the ratio of compressional to shear velocities under loading.

The experimental data underlying Zhurkov's formula (0.1) are also reproduced in numerical simulations.

The results of experiments for a tensile load σ are shown in Fig.4 and are taken from (author?) [Regel 1972]:

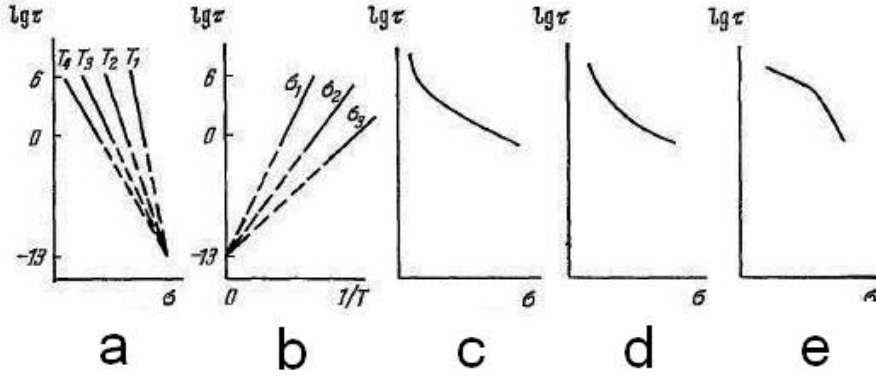


Fig.4

- (a) $\ln \tau$ as a function of σ at constant T ,
- (b) $\ln \tau$ as a function of T^{-1} at constant σ ,
- (c)-(e) deviations of $\ln \tau$ as a function of σ from linearity.

The deviations from linearity are explained by some "complicating factors".

In the conditions of two previous sections we get from (5.7) the equation

$$\frac{dp}{dt} = (1-p) c_1 \exp \left\{ \beta \left[a_0 \left(\frac{\sigma}{1-\theta p} \right)^2 - H \right] \right\} - p c_0 \exp \{-\beta U\}.$$

We assume that the lifetime τ of a specimen under tensile load σ at temperature T is the time for which the density of microcracks p changes from 0 to the critical value p_0 (e.g. $p_0 = 0.2$).

For different sets of constants there is a sufficiently large domain of σ and T where $\ln \tau$ as a function of σ and T plot as straight lines similar to experimental data as shown in Fig.5. This fact allows us to present this numeral result in the Zhurkov form (0.1).

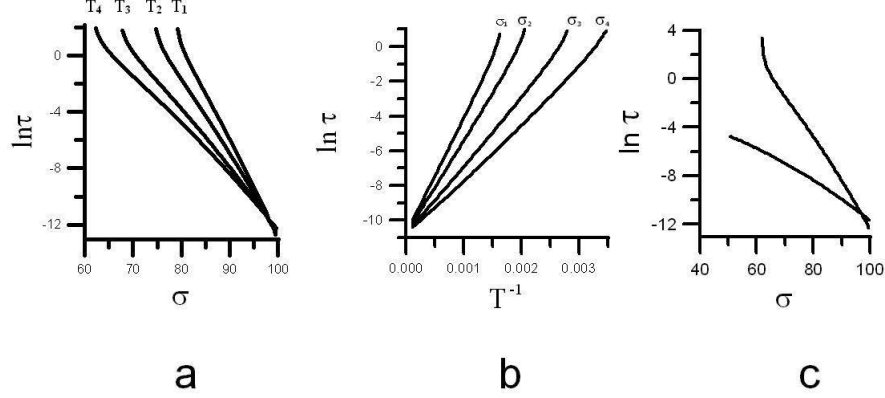


Fig.5

- (a) $\ln \tau$ as a function of σ at constant T ,
- (b) $\ln \tau$ as a function of T^{-1} at constant σ ,
- (c)-(e) deviations of $\ln \tau$ as a function of σ from linearity.

As the linearity and the deviations from linearity are present in the solutions of the equation, the need to introduce “complicating factors” vanishes.

Appendix

ASYMPTOTIC STATIONARY STATE OF THE SIMPLEST MEAN FIELD MARKOV PROCESS

Definition. Let $\xi_N(t) \equiv \{\xi_N(x, t), x \in \Omega_N\}$, $\xi_N(x, t) = 0, 1$, $|\Omega_N| = N$, $t \geq 0$, ($|A|$ denotes a number of elements in A) be N -component continuous-time Markov process with state-space $\{0, 1\}^{\Omega_N}$. Denote $n(t) = n(\xi_N(t)) = |\{x \in \Omega_N : \xi_N(x, t) = 1\}|$ and let $\Lambda(y)$, $M(y)$ be positive continuous functions on $[0, 1]$. We assume that only one point spin-flip translations $\{0 \rightarrow 1, 1 \rightarrow 0\}$ are possible, with the rates given by conditional probabilities

$$\begin{cases} \Pr\{\xi_N(x, t+h) = 1 | \xi_N(x, t) = 0, n(t)\} = \Lambda\left(\frac{n(t)}{N}\right) h + o(h), \\ \Pr\{\xi_N(x, t+h) = 0 | \xi_N(x, t) = 1, n(t)\} = M\left(\frac{n(t)}{N}\right) h + o(h) \end{cases} \quad (A.1)$$

This process will be named “*mean field*” Markov process. The random process $n(t)$ is also Markov process. For it

$$\begin{cases} \Pr\{n(t+h) = k+1 | n(t) = k\} = (N-k)\Lambda\left(\frac{k}{N}\right) h + o(h), \\ \Pr\{n(t+h) = k-1 | n(t) = k\} = kM\left(\frac{k}{N}\right) h + o(h), \\ \Pr\{n(t+h) = k | n(t) = k\} = 1 - (N-k)\Lambda\left(\frac{k}{N}\right) h - kM\left(\frac{k}{N}\right) h + o(h) \end{cases}$$

So for $m(t) = \frac{n(t)}{N}$ we have

$$\frac{d \langle m(t) \rangle}{dt} = \langle (1 - m(t))\Lambda(m(t)) - m(t)M(m(t)) \rangle,$$

where $\langle \dots \rangle$ is the mathematical expectation. It is known ((**author?**) [Ethier 1986], [(**author?**) [Malyshev 2008]]) that if $N \rightarrow \infty$ and $m(0)$ tends to a nonrandom limit $p(0)$, then $m(t)$ converges in probability to $p(t)$ and $p(t)$ is described by equation

$$\frac{dp(t)}{dt} = [1 - p(t)] \Lambda(p(t)) - p(t)M(p(t))$$

It follows from this that if random variables $\xi_N(x, 0)$ are independent and equally distributed in an initial time moment, i.e. they have the Bernoulli distribution with the parameter $p(0)$, than in the limit $N \rightarrow \infty$ the joint distribution of values $\xi_N(x, t)$ for any t and any finite set of x converges to the Bernoulli distribution with the parameter $p(t)$. It is based on the exchangeability of these random variables.

Below we'll prove that for the case of the absence of phase transitions the components $\xi_N(x)$ in the stationary state are asymptotically independent in the thermodynamic limit $N \rightarrow \infty$.

Let $A \subseteq \Omega_N$, and

$$P_N(A, t) = \Pr\{\xi_N(x, t) = 1, x \in A, \xi_N(x, t) = 0, x \in \Omega_N \setminus A\}.$$

It follows from (A.1) that

$$\begin{aligned} \frac{dP_N(A, t)}{dt} &= \sum_{x \in A} P_N(A \setminus \{x\}, t) \Lambda\left(\frac{|A| - 1}{N}\right) - |A| P_N(A, t) M\left(\frac{|A|}{N}\right) + \\ &+ \sum_{x \in \Omega_N \setminus A} P_N(A \cup \{x\}, t) M\left(\frac{|A| + 1}{N}\right) - (N - |A|) P_N(A, t) \Lambda\left(\frac{|A|}{N}\right). \quad (A.2) \end{aligned}$$

Theorem. *If $\Lambda(y)$, $M(y)$ are strictly positive and differentiable, $f(y) = \frac{\Lambda(y)}{M(y)}$, function*

$$F(y) = \int_0^y \ln f(z) dz - y \ln y - (1 - y) \ln(1 - y)$$

as a single maximum in p , $p \in (0, 1)$, (the absence of phase transitions) and $F''(p)$ is negative, then in the stationary state $\xi(x)$ are independent in the limit $N \rightarrow \infty$ and $p = \lim_{N \rightarrow \infty} E_N \left\{ \frac{n(\xi_N)}{N} \right\}$ is the solution of the equation

$$p = \frac{f(p)}{1 + f(p)}.$$

Proof. Let's set

$$f_N(y) = \frac{\Lambda(y)}{M(y + \frac{1}{N})}.$$

By means of direct substitution in (A.2) it is easy to check that the solution of the system

$$\frac{dP_N(A, t)}{dt} = 0$$

has a form □

$$\begin{aligned} P_N(A) &= \frac{1}{Z_N} \exp \left\{ \sum_{m=0}^{|A|-1} \ln f_N \left(\frac{m}{N} \right) \right\}, \\ Z_N &= \sum_{A \subseteq \Omega_N} \exp \left\{ \sum_{m=0}^{|A|-1} \ln f_N \left(\frac{m}{N} \right) \right\} = \\ &= \sum_{|A|=0}^N \frac{N!}{(N-|A|)!|A|!} \exp \left\{ \sum_{m=0}^{|A|-1} \ln f_N \left(\frac{m}{N} \right) \right\}. \end{aligned}$$

We have for any differentiable function g

$$\begin{aligned} \sum_{m=0}^{M-1} g\left(\frac{m}{N}\right) &= N \sum_{m=0}^M g\left(\frac{m}{N}\right) \frac{1}{N} - g\left(\frac{M}{N}\right) = \frac{g(0) - g(\frac{M}{N})}{2} + \\ &+ N \sum_{m=1}^M \frac{g(\frac{m-1}{N}) + g(\frac{m}{N})}{2} \frac{1}{N} = \\ &= \frac{g(0) - g(\frac{M}{N})}{2} + N \int_0^{\frac{M}{N}} g(z) dz + O\left(\frac{1}{N}\right). \end{aligned}$$

Using

$$\ln f_N(y) \sim \ln f(y) - \frac{1}{N} \frac{d}{dy} \ln M(y)$$

we have

$$\sum_{m=0}^{|A|-1} \ln f_N \left(\frac{m}{N} \right) \sim N \int_0^y \ln f(z) dz + \frac{\ln \Lambda(0)M(0) - \ln \Lambda(y)M(y)}{2},$$

where $y = \frac{|A|}{N}$.

So

$$P_N(A) \sim \frac{1}{Z_N \sqrt{\Lambda(y)M(y)}} \exp \left\{ N \int_0^y \ln f(z) dz \right\}. \quad (A.3)$$

Using Stirling's formula Z_N we have for the partition function

$$Z_N \sim \int_0^1 \sqrt{\frac{1}{2\pi N(1-y)y\Lambda(y)M(y)}} \exp \{ NF(y) \} dy. \quad (A.4)$$

For the probabilities $P_{A,N}(B) = \Pr_N \{ \xi_N(x, t) = 1, x \in B, \xi_N(x, t) = 0, x \in A \setminus B \}$, $B \subseteq A \subset \Omega_N$, we have

$$P_{A,N}(B) = \frac{1}{Z_N} \sum_{B \leq n \leq N-|A|+|B|} \frac{(N-|A|)!}{[N-|A|-n+|B|]!(n-|B|)!} \exp \left\{ \sum_{m=0}^{n-1} \ln f_N \left(\frac{m}{N} \right) \right\}.$$

Now

$$\frac{(N-|A|)!}{[N-|A|-n+|B|]!(n-|B|)!} \sim (1-y)^{|A|-|B|} y^{|B|} \frac{N!}{(N-n)!n!}, \quad y = \frac{n}{N},$$

and again using Stirling's formula we have

$$P_{A,N}(B) \sim \frac{1}{Z_N} \int_{\frac{|B|}{N} \leq y \leq 1 - \frac{|A|-|B|}{N}} (1-y)^{|A|-|B|} y^{|B|} \frac{\exp \{ NF(y) \}}{\sqrt{2\pi N(1-y)y\Lambda(y)M(y)}} dy.$$

By using the Laplace's method we have

$$\begin{aligned} & \int_{\frac{|B|}{N} \leq y \leq \frac{|A|-|B|}{N}} (1-y)^{|A|-|B|} y^{|B|} \frac{\exp \{ NF(y) \}}{\sqrt{2\pi N(1-y)y\Lambda(y)M(y)}} dy \sim \\ & \sim (1-p)^{|A|-|B|} p^{|B|} \frac{\exp \{ NF(p) \}}{N \sqrt{F''(p)(1-p)p\Lambda(p)M(p)}} \end{aligned}$$

and

$$Z_N \sim \frac{\exp \{ NF(p) \}}{N \sqrt{F''(p)(1-p)p\Lambda(p)M(p)}}.$$

Therefore

$$P_{A,N}(B) \sim (1-p)^{|A|-|B|} p^{|B|}$$

proves the asymptotic independence.

From

$$F'(p) = \ln f(p) - \ln \frac{p}{1-p} = 0$$

it follows that

$$p = \frac{f(p)}{1 + f(p)}.$$

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References

- [Day 1972] Day U. A., The Thermodynamics of Simple Materials with Fading Memory, Springer-Verlag, Berlin, NewYork, Heidelberg, 1972.
- [Gertzik 1998] Gertzik V.M., Gibbsian ensemble of noninteracting cracks. Numerical Modelling and Analysis of Geophysical Processes, Computational Seismology, 20, 71-81, Allerton, 1988 (in russian).
- [Gertzik 1994] Gertzik V.M., On the mechanical properties of model solid in the activation theory. Geodynamics and Earthquake Prediction. Computational Seismology, 26, 58-67, M.: Nauka, 1994 (in russian).
- [Gutenberg 1954] Gutenberg B. and C.F. Richter. Seismicity of the Earth and Associated Phenomena, 2nd ed. Princeton, N.J.: Princeton University Press, pages 17–19 (1954)
- [Ethier 1986] Ethier S.N., Kurtz T.G. Markov processes: characterization and convergence, John Wiley & Sons. 1986..
- [Frenkel 1946] Frenkel, J., Kinetic Theory of Liquids, Oxford, Clarendon Press, 1946.
- [Kac 1968] Kac M., Mathematical mechanisms of phase transitions, Statistical Physics, Phase Transitions and Superfluidity, Vol.1, Brandeis University Summer Institute in Theoretical Physics, 1966 (ed. by M. Chrétien, E.P. Gross and S. Deser), 241-305, Gordon and Breach, (1968).
- [Kasahara 1981] Kasahara, K. (1981) Earthquake Mechanics, Cambridge University Press.

- [Landau 1986] L. D. Landau and E. M. Lifshitz, Theory of Elasticity, Pergamon Press, 1986.
- [Malyshev 2008] Malyshev V A. and Pirogov S A Reversibility and irreversibility in stochastic chemical kinetics, Russ. Math. Surv. 2008, p.63 16
- [Regel 1972] Regel V.R., Slutsker A.I., Tomashevskii A.E., Kinetic nature of the strength of solids. Sov. Phys Uspehi, 1972, 15 (1), 45–65.
- [Zhurkov 1965] S. N. Kinetic concept of the strength of solids. Int. J. Fract. Mech. 1 (1965): 311-322